Time-Dependent Orthogonal Polynomials and Theory of Soliton

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Abstract

By introducing a time variable to the theory of orthogonal polynomials, it is shown that the matrix models of two-dimensional gravity, the six vertex model of two-dimensional lattice statistics and the random matrix theory of level statistics are all described by the theory of soliton, *i.e.* Toda molecule equation.

1 Introduction

Theory of orthogonal polynomials has many applications in diverse branches of theoretical physics. Recently some remarkable applications[1-3] have been found under the perspective of completely integrable system (theory of soliton). Here I wish to describe the outline of ref.3 which claims that a theory of *time-dependent* orthogonal polynomials (TDOP) interrelates 1) the matrix model of two-dimensional gravity, 2) the six vertex model of two-dimensional lattice statistics, 3) the random matrix theory of level statistics and 4) the soliton theory of Toda molecule equation (see Fig.1).

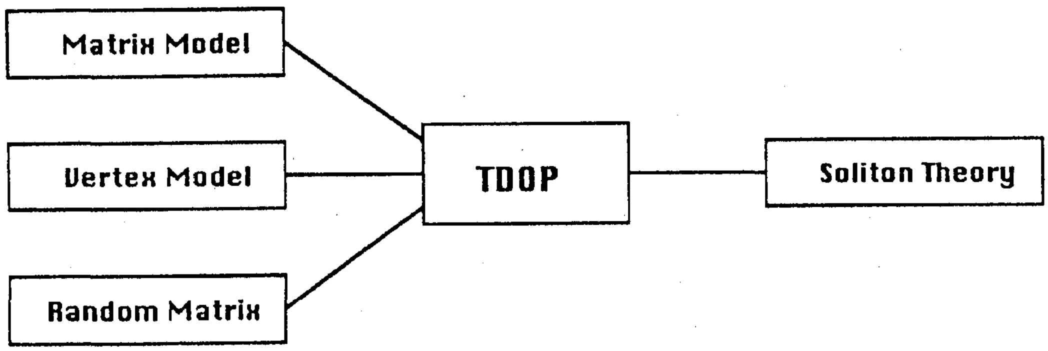

Fig. 1. Time-dependent orthogonal polynomials (TDOP) interrelate all theory.

The most remarkable fact found here is that *in these theories commonly appears the Toda molecule equation*, which is defined by (in Hirota form)

$$\tau_n \tau_n'' - \tau_n'^2 = \tau_{n+1}\tau_{n-1}, \tag{1}$$

where the prime denotes time derivative.

2 Time-Dependent Orthogonal Polynomials and Toda Molecule Equation

The distribution function $\rho(\lambda;t)$, or the weight function $w(\lambda;t) = \mathrm{d}\rho(\lambda;t)/\mathrm{d}\lambda$, defines uniquely the ortho-normal polynomial $\psi_n(\lambda;t)$ according to Hilbert-Schmidt's diagonalization method.

$$\psi_n(\lambda;t) = \frac{1}{\sqrt{\tau_n\tau_{n-1}}}\begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & \lambda & \dots & \lambda^n \end{vmatrix}, \tag{2}$$

where s_n's are the moments defined by $s_n(t) = \int \lambda^n \mathrm{d}\rho(\lambda;t)$ and the tau function τ_n is defined by

$$\tau_n(t) = \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & \dots & s_{2n} \end{vmatrix}, \tag{3}$$

which is a Hankel-Hadamard determinant.

Here we assume further that

$$s_n(t) = \frac{\mathrm{d}^n s_0(t)}{\mathrm{d}t^n} \tag{4}$$

and call this the derivative Hankel property. Then by using Laplace-Jacobi theorem, we can prove that the tau function $\tau_n(t)$ satisfies the Toda molecule equation given by eq.(1). The Toda molecule equation is a soliton system whose Lax pair is given by

$$a_{n+1}\psi_{n+1} + b_n\psi_n + a_n\psi_{n-1} = \lambda\psi_n, \quad \frac{\mathrm{d}\psi_n}{\mathrm{d}t} = -\frac{1}{2}b_n\psi_n - a_n\psi_{n-1}. \tag{5}$$

The first equation is known as the scattering problem in soliton theory, and also as the three-term relation in orthogonal polynomial theory.

From the compatibility condition of eq.(5), the equations of motion for field variables a_n, b_n are given by

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \frac{1}{2}a_n(b_n - b_{n-1}), \quad \frac{\mathrm{d}b_n}{\mathrm{d}t} = a_{n+1}^2 - a_n^2. \tag{6}$$

The relation between field variables and tau function are expressed by

$$a_{n+1}^2 = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \quad b_n = \frac{\tau_n'}{\tau_n} - \frac{\tau_{n-1}'}{\tau_{n-1}}. \tag{7}$$

Toda molecule equation (1) is derived by combining eqs.(6) and (7).

We can summarize the above results as follows. If the distribution function $\rho(\lambda; t)$ has the derivative Hankel property (eq.(4)), then the tau function (eq.(3)) obeys the Toda molecule equation (eq.(1)). Inversely, $\tau_0(t) \equiv s_0(t)$ determines the distribution function $\rho(\lambda; t)$ uniquely.

3 Applications of the TDOP Theory

(1) Matrix model of two-dimensional gravity

The first example is from matrix model of two-dimensional gravity. We recognize that the derivative Hankel property is satisfied if we assume

$$\frac{\mathrm{d}\rho(\lambda;t)}{\mathrm{d}\lambda} = \exp\left(-U(\lambda) + \lambda t\right). \tag{8}$$

Now by using eq.(8), we can derive the following identities:

$$n = a_n \int \psi_n \frac{\mathrm{d}U}{\mathrm{d}\lambda} \psi_{n-1} \mathrm{d}\rho, \quad t = \int \psi_n^2 \frac{\mathrm{d}U}{\mathrm{d}\lambda} \mathrm{d}\rho. \tag{9}$$

Therefore, if we set further the potential $U(\lambda) = \frac{1}{2}g_2\lambda^2 + \frac{1}{4}g_4\lambda^4$ (the pure gravity case), we obtain

$$\begin{aligned} n &= a_n^2\{g_2 + g_4(a_n^2 + a_{n+1}^2 + a_{n-1}^2 + b_n^2 + b_n b_{n-1} + b_{n-1}^2)\}, \\ t &= g_2 b_n + g_4\{b_{n+1}a_{n+1}^2 + b_n(2a_{n+1}^2 + b_n^2 + 2a_n^2) + b_{n-1}a_n^2\}. \end{aligned} \tag{10}$$

These are an extension of the first-kind discrete Painlevé equation to include two variables: one (time t) is continuous and the other (space n) is discrete. Other discrete Painlevé equations are also derived by changing the potential.

(2) Izergin's six vertex model

The second example is from lattice statistical model. According to Izergin a special case of the six vertex model on $N \times N$ lattice has the partition function

$$Z_N = (2w)^N((w_4 + w_3)(w_4 - w_3))^{N^2}(\textstyle\prod_{k=1}^{N-1} k!)^{-2}\det(B), \tag{11}$$

where $w_4 + w_3 = \sinh(t+\gamma)$, $w_4 - w_3 = \sinh(t-\gamma)$, $2w = \sinh(2\gamma)$ and the matrix element of $N \times N$ matrix B is defined by $B_{ij} = \mathrm{d}^{i+j-2}s_0(t)/\mathrm{d}t^{i+j-2}$,

$s_0(t) = 1/(\sinh(t+\gamma)\sinh(t-\gamma))$. Now we easily recognize that the determinant of B is of derivative Hankel type, and therefore is given by the tau function $\tau_{N-1}(t)$. Consequently this tau function satisfies the Toda molecule equation as has been discussed. Thus we can conclude that Izergin's six vertex model is also described by soliton equation.

(3) Random matrix theory of level statistics

The last example is from the random matrix theory of level statistics. It is easily noted that eq.(3) is rewritten as

$$\tau_n(t) = \frac{1}{(n+1)!}\int\cdots\int \prod_{i=1}^{n+1} \mathrm{d}\rho(\lambda_i;t)\prod_{j<k}^{n+1}(\lambda_j-\lambda_k)^2. \tag{12}$$

Therefore the normalized distribution function of energy levels $\lambda_1, ..., \lambda_N$ is expressed by

$$P_N(\lambda_1, ..., \lambda_N; t) = \frac{1}{\tau_{N-1}(t)N!}\prod_{k=1}^{N} w(\lambda_k;t)\prod_{i<j}^{N}(\lambda_i-\lambda_j)^2, \tag{13}$$

In other words, the present theory corresponds to the *unitary* ensemble of random matrix theory. This approach of time-dependent random matrix theory may be regarded as a dynamical theory of the quantum chaos. In ref.3, Brownian motion model of level statistics formulated by Dyson is discussed according to the present approach.

4 Prospects of the TDOP Theory

The present theory of time-dependent orthogonal polynomials posseses a discrete time analog. Such extension derives the difference-difference Toda molecule equation

$$\tau_n(\ell)^2 - \tau_n(\ell-1)\tau_n(\ell+1) + \tau_{n+1}(\ell-1)\tau_{n-1}(\ell+1) = 0, \tag{14}$$

which also has a Hankel determinant solution.[2)]

The present theory has also a generalization of *q-deformation* type (the quantum group). In such theories we will encounter many classical q-polynomials proposed by Heine, Askey, Wilson and others. The relationship among these subjects is summarized in Fig.2.

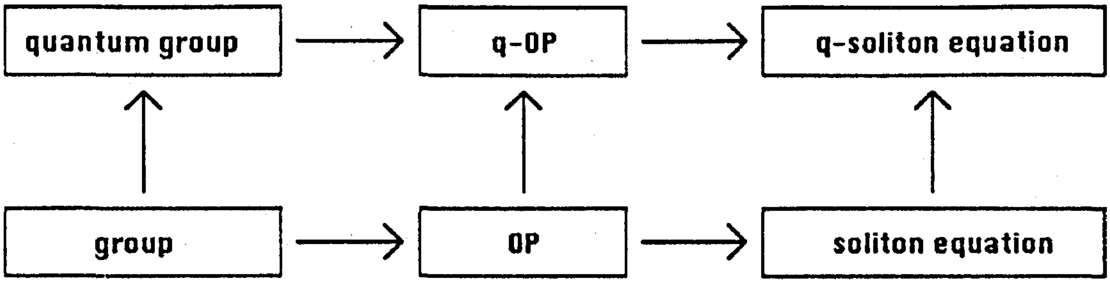

Fig. 2. Quantum group, q-orthogonal polynomials and q-soliton equations.

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Further references can be found in this paper.